On the Dimension of the Set of Two-View Multi-Homography Matrices

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Abstract. It is shown that the set of all multi-homography matrices describing I-element families of interdependent homographies between two views has dimension $4I + 7$.

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1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^{m\times n}$ denote the set of $m \times n$ matrices with entries in $\mathbb{R}$. We identify coordinate vectors in $\mathbb{R}^n$ with $n \times 1$ matrices in $\mathbb{R}^{n\times 1}$, or, what is the same, with length-n column vectors with real entries. Given $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{b} \in \mathbb{R}^3$, $\{\mathbf{v}_i\}_{i=1}^I \subset \mathbb{R}^3$ and $\{w_i\}_{i=1}^I \subset \mathbb{R}$, let, for every $i = 1, \dots, I$, $\mathbf{H}_i$ be the 3×3 matrix defined by

$$\mathbf{H}_i = w_i\mathbf{A} + \mathbf{b}\mathbf{v}_i^\top,$$

where the superscript $\top$ denotes transposition. As it turns out (see below), each $\mathbf{H}_i$, provided that it is invertible, is a *homography matrix* for a homography of specific geometric significance, acting in two-dimensional real projective space. For each $i = 1, \dots, I$, let $\mathbf{h}_i = \text{vec}(\mathbf{H}_i)$, where vec denotes column-wise vectorisation [9], and let $\mathbf{H}$ be the $9 \times I$ matrix given by

$$\mathbf{H} = [\mathbf{h}_1, \dots, \mathbf{h}_I].$$

Henceforth any $\mathbf{H} = \mathbf{H}(\mathbf{A}, \mathbf{b}, \mathbf{v}_1, \dots, \mathbf{v}_I, w_1, \dots, w_I)$ of this form, irrespective of whether the underlying constituent matrices $\mathbf{H}_i$ are invertible or not, will be referred to as a *two-view multi-homography matrix*, or simply as a *multi-homography matrix*. The set of all multi-homography matrices will be denoted by $\mathcal{H}$. The present paper addresses the problem of the computation of the *dimension* of $\mathcal{H}$. The notion of dimension that is of relevance here has to do with the fact $\mathcal{H}$ is a polynomial image of $\mathbb{R}^{4I+12}$. Recall that a map $\mathbf{f} = [f_1, \dots, f_n]^\mathsf{T} \colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be *polynomial* if the functions $f_i = f_i(\mathbf{x})$ are polynomial functions in the entries of the vector argument $\mathbf{x} = [x_1, \dots, x_m]^\mathsf{T}$. The celebrated Tarski–Seidenberg theorem [1, 2] ensures that the image of any polynomial map $\mathbf{f} \colon \mathbb{R}^m \to \mathbb{R}^n$ is a semi-algebraic set—that is, a finite union of sets, each defined by a finite conjunction of polynomial equalities and inequalities with real coefficients. Any semi-algebraic set is locally a submanifold on a dense open subset. This permits defining the dimension of a semi-algebraic set to be the largest dimension at points around which the set is a submanifold.

The present paper reveals that the dimension of the semi-algebraic set $\mathcal{H}$ is equal to $4I + 7$. This result has its origins in computer vision in the context of solving certain statistical parameter estimation problems [3–5]. One issue that arises naturally in connection with these problems is the question of characterising the Zariski closure of $\mathcal{H}$, which is the smallest set containing $\mathcal{H}$ defined by finitely many polynomials with real coefficients, as a set of points satisfying explicit constraints put on the ambient Euclidean space. While some constraints—like the so-called *rank-four constraint* (to be discussed later)—have been identified, a full set of constraints has not been found yet. It is hoped that the dimensionality result established here will facilitate the task of uncovering a complete set of relevant constraints.

2. Geometric link

We start by explaining the geometric meaning of the matrices introduced in the Introduction.

Recall that if V is a vector space, then the *projective space* $P(V)$ of V is the set of one-dimensional vector subspaces of V. We write $P(\mathbb{R}^{n+1})$ as $P^n(\mathbb{R})$. Any one-dimensional subspace of $P^n(\mathbb{R})$ is the set of all multiples of a non-zero vector in $\mathbb{R}^{n+1}$. Given $\mathbf{x} = [x_1, \dots, x_{n+1}]^\mathsf{T} \in \mathbb{R}^{n+1} \setminus \{0\}$, let $[\mathbf{x}] \in P^n(\mathbb{R})$ be the set of all multiples of $\mathbf{x}$. Then $\mathbf{x}$ is said to be a *representative vector* for $[\mathbf{x}]$. If $\rho \neq 0$, then $\rho\mathbf{x}$ is another representative vector for $[\mathbf{x}]$ so that $[\mathbf{x}] = [\rho\mathbf{x}]$. Any member $\underline{\mathbf{x}} = [x_1, \dots, x_n]^\mathsf{T}$ of $\mathbb{R}^n$ can be identified with the point $[\mathbf{x}]$ in $P^n(\mathbb{R})$ with $\mathbf{x} = [x_1, \dots, x_n, 1]^\mathsf{T}$; the vector $\mathbf{x}$ is then called the *homogeneous vector* for $\underline{\mathbf{x}}$. The part of $P^n(\mathbb{R})$ identified with $\mathbb{R}^n$ consists of the so-called *ordinary points* of $P^n(\mathbb{R})$, the remaining part $P^n(\mathbb{R}) \setminus \mathbb{R}^n$ being comprised of the so-called *ideal points* of $P^n(\mathbb{R})$.

Given a linear map $\mathbf{A}$, let $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ denote the *range space* and the *null space* of $\mathbf{A}$, respectively. For a matrix $\mathbf{A}$, let $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ denote the *column space* (or the *range*) and the *column null space* (or the *kernel*) of $\mathbf{A}$, respectively.

If $\mathbf{H}$ is an $(n+1) \times (n+1)$ invertible matrix, then $\mathbf{H}$ gives rise to a *homography* $P(\mathbf{H}) \colon P^n(\mathbb{R}) \to P^n(\mathbb{R})$ given by

$$P(\mathbf{H})([\mathbf{x}]) = [\mathbf{H}\mathbf{x}], \quad \mathbf{x} \in \mathbb{R}^{n+1}.$$

If $\rho \neq 0$, then $\rho\mathbf{H}$ and $\mathbf{H}$ define the same homography, and any matrix of the form $\rho\mathbf{H}$ is a *homography matrix* for $P(\mathbf{H})$. If $\mathbf{P}$ is an $(n+1) \times (m+1)$

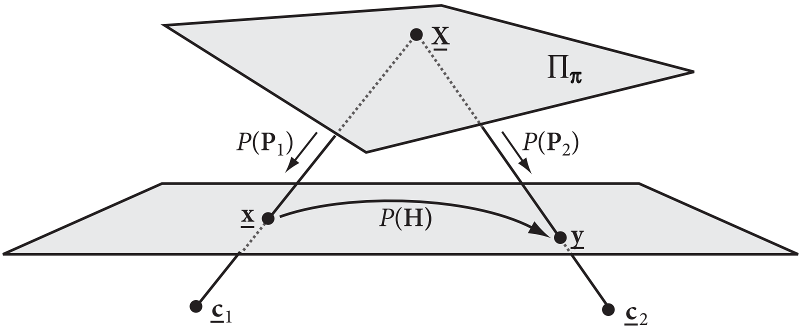

FIGURE 1. Homography between two views induced by a plane.

matrix with $n < m$ and of rank $n+1$, then $C = P(\mathcal{N}(\mathbf{P}))$ is a projective subspace of $P^m(\mathbb{R})$ of dimension $m-n-1$ and $\mathbf{P}$ gives rise to a *projection* $P(\mathbf{P})\colon P^m(\mathbb{R}) \setminus C \to P^n(\mathbb{R})$ *from the centre* C given by

$$P(\mathbf{P})([\mathbf{x}]) = [\mathbf{P}\mathbf{x}], \quad \mathbf{x} \in \mathbb{R}^{m+1}.$$

If $\rho \neq 0$, then $\rho\mathbf{P}$ and $\mathbf{P}$ define the same projection, and any matrix of the form $\rho\mathbf{P}$ is a *projection matrix* for $P(\mathbf{P})$.

Any non-zero vector $\boldsymbol{\pi} \in \mathbb{R}^{n+1}$ defines the hyperplane in $P^n(\mathbb{R})$

$$\Pi_{\boldsymbol{\pi}} = \{[\mathbf{x}] \in P^n(\mathbb{R}) \mid \boldsymbol{\pi}^\mathsf{T}\mathbf{x} = 0\},$$

with all non-zero multiples of $\boldsymbol{\pi}$ defining the same hyperplane.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two 3×4 matrices given by

$$\mathbf{P}_1 = [\mathbf{I}_3, \mathbf{0}] \quad \text{and} \quad \mathbf{P}_2 = [\mathbf{A}, -\mathbf{b}],$$

where $\mathbf{I}_3$ is the 3×3 identity matrix, $\mathbf{0}$ is the length-3 zero vector, and $\mathbf{A} \in \mathbb{R}^{3\times 3}$ and $\mathbf{b} \in \mathbb{R}^3$ are such that $\mathbf{P}_2$ has rank 3. The matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ give rise to two projections $P(\mathbf{P}_1)\colon P^3(\mathbb{R}) \to P^2(\mathbb{R})$ and $P(\mathbf{P}_2)\colon P^3(\mathbb{R}) \to P^2(\mathbb{R})$ with zero-dimensional (point) centres $C_1 \in P^3(\mathbb{R})$ and $C_2 \in P^3(\mathbb{R})$. The centre C_1 actually lies in $\mathbb{R}^3$ and is represented by the vector $\underline{\mathbf{c}}_1 = [0,0,0]^\mathsf{T}$. Suppose that the other centre also lies in $\mathbb{R}^3$ and is represented by a length-3 vector $\underline{\mathbf{c}}_2$. Let $\boldsymbol{\pi} = [\mathbf{v}^\mathsf{T}, w]^\mathsf{T}$ be a length-4 vector with $\mathbf{v} \in \mathbb{R}^3$ and $w \in \mathbb{R}$, and let $\Pi_{\boldsymbol{\pi}}$ be the corresponding plane in $P^3(\mathbb{R})$. Then, associated with $P(\mathbf{P}_1)$, $P(\mathbf{P}_2)$, and $\Pi_{\boldsymbol{\pi}}$, there is a specific homography acting in $P^2(\mathbb{R})$. The action of this homography on the ordinary points of $P^2(\mathbb{R})$ can be described as follows. Given $\underline{\mathbf{x}} \in \mathbb{R}^2 \subset P^2(\mathbb{R})$, issue a line through $\underline{\mathbf{c}}_1$ and $\underline{\mathbf{x}}$ and let $\underline{\mathbf{X}}$ be the point of intersection of this line and $\Pi_{\boldsymbol{\pi}}$. Next issue a line through $\underline{\mathbf{X}}$ and $\underline{\mathbf{c}}_2$ and let $\underline{\mathbf{y}}$ be the point of intersection of this line and $\mathbb{R}^2$. The mapping that takes $\underline{\mathbf{x}}$ to $\underline{\mathbf{y}}$ is the homography in question (see Figure 1). It can be shown that this homography can be represented as $P(\mathbf{H})$ with

$$\mathbf{H} = w\mathbf{A} + \mathbf{b}\mathbf{v}^\mathsf{T};$$

in other words, if $\underline{x}$ and $\underline{y}$ are represented by respective homogeneous vectors $\mathbf{x}$ and $\mathbf{y}$, then

$$[\mathbf{y}] = [\mathbf{H}\mathbf{x}]$$

(see [8]). The mapping $P(\mathbf{H})$ is termed the homography induced by the plane $\Pi_{\boldsymbol{\pi}}$ between the views described by $P(\mathbf{P}_1)$ and $P(\mathbf{P}_2)$.

With $\mathbf{P}_1$ and $\mathbf{P}_2$ as above, if $\{\boldsymbol{\pi}_i\}_{i=1}^{I}$ is a set of length-4 vectors $\boldsymbol{\pi}_i = [\mathbf{v}_i^\top, w_i]^\top$ with $\mathbf{v}_i \in \mathbb{R}^3$ and $w_i \in \mathbb{R}$, then, for each $i = 1, \ldots, I$, the i-th plane $\Pi_{\boldsymbol{\pi}_i}$ induces a homography $P(\mathbf{H}_i)$ with

$$\mathbf{H}_i = w_i \mathbf{A} + \mathbf{b}\mathbf{v}_i^\top.$$

These homographies are all interlinked, as they are all generated under the common views described by $P(\mathbf{P}_1)$ and $P(\mathbf{P}_2)$.

3. Algebro-geometric prerequisites

Let $\mathbb{R}[x_1, \ldots, x_n]$ denote the set of all polynomials in the indeterminates $x_1, \ldots, x_n$ with real coefficients. A subset V of $\mathbb{R}^n$ is a *variety* or an *algebraic set* if there exist polynomials $p_1, \ldots, p_m$ in $\mathbb{R}[x_1, \ldots, x_n]$ such that

$$V = \mathbb{V}(p_1, \ldots, p_m),$$

where

$$\mathbb{V}(p_1, \ldots, p_m) = \{\mathbf{x} \in \mathbb{R}^n \mid p_\mu(\mathbf{x}) = 0 \quad \text{for all } 1 \leq \mu \leq m\}.$$

A subset S of $\mathbb{R}^n$ is a *semi-algebraic set* if

$$S = \bigcup_{\mu=1}^{m} \bigcap_{\nu=1}^{n_\mu} \{\mathbf{x} \in \mathbb{R}^n \mid p_{\mu,\nu}(\mathbf{x}) \rhd_{\mu\nu} 0\},$$

where $p_{\mu,\nu}$ are polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ and $\rhd_{\mu\nu}$ is one of the three relational operators $<$, $=$, $>$. In other words, a semi-algebraic set is a finite union of sets, each determined by a finite number of polynomial equations and inequalities with real coefficients.

A map $\mathbf{f} \colon S \to T$, where $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ are semi-algebraic sets, is *semi-algebraic* if the graph of $\mathbf{f}$,

$$\{[\mathbf{x}^\top, \mathbf{f}(\mathbf{x})^\top]^\top \in \mathbb{R}^{n+m} \mid \mathbf{x} \in S\},$$

is a semi-algebraic subset of $\mathbb{R}^{n+m}$. If $\mathbf{f} = [f_1, \ldots, f_m]^\top$ is a polynomial map, then $\mathbf{f}$ is semi-algebraic because its graph can be described by m polynomial equalities

$$y_\mu - f_\mu(\mathbf{x}) = 0 \quad (1 \leq \mu \leq m).$$

A key result about semi-algebraic sets is the Tarski–Seidenberg theorem saying that if $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ are semi-algebraic sets and $\mathbf{f} \colon S \to T$ is a semi-algebraic map, then the image $\mathbf{f}(S) \subset T$ is a semi-algebraic set [1,2]. In particular, the images of polynomial maps are semi-algebraic.

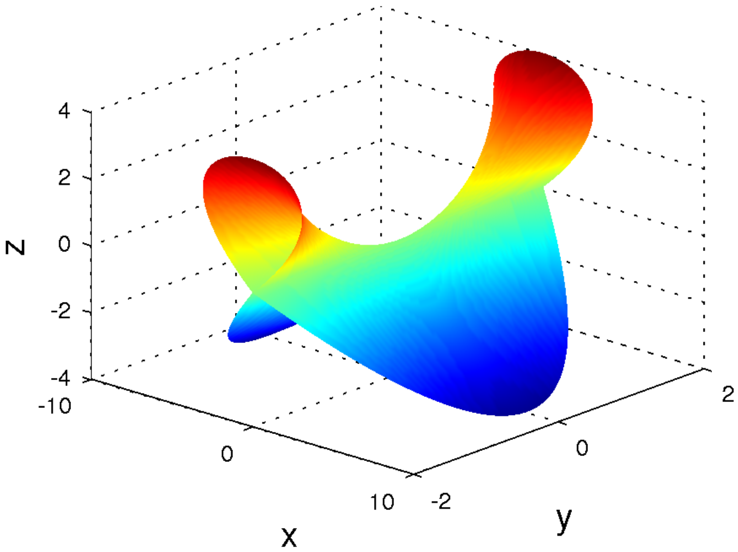

FIGURE 2. Plot of a portion of the variety $\mathbb{V}(x^2 - y^2z^2 + z^3)$.

Some semi-algebraic sets are smooth manifolds and some are not. Consider, for example, the image in $\mathbb{R}^3$ of $\mathbb{R}^2$ by the polynomial map

$$(t, u) \mapsto (t(u^2 - t^2), u, u^2 - t^2).$$

It coincides with the variety $\mathbb{V}(x^2 - y^2z^2 + z^3)$. This variety is not a smooth manifold because, locally, at each point of the y-axis other than the origin, the surface looks like the intersection of two smooth manifolds—see Figure 2.

While not all semi-algebraic sets are manifolds, it turns out that every semi-algebraic set can be meaningfully assigned a dimension. This is a consequence of the fact that every semi-algebraic set admits a *stratification*. To get an idea of the concept, consider again the variety $\mathbb{V}(x^2 - y^2z^2 + z^3)$. This variety can be represented as the set-theoretic union of several two-dimensional surfaces together with a one-dimensional smooth manifold, the y-axis. These smooth manifolds constitute a stratification of $\mathbb{V}(x^2-y^2z^2+z^3)$.

Formally, a stratification of a set $X \subset \mathbb{R}^n$ is a finite partition $\{X_i\}_{i\in I}$ of X such that

(S1) each X_i, called a *stratum* of X, is a d_i-dimensional smooth manifold in $\mathbb{R}^n$;

(S2) **(frontier condition)** if $X_j \cap \overline{X_i} \neq \emptyset$, then $X_j \subset \overline{X_i}$ and $d_j < d_i$,[1] where $\overline{Y}$ denotes the closure of Y.

A stratification is called *semi-algebraic* if every stratum is semi-algebraic. A *stratified set* is a set that admits a stratification. The dimension of a stratified

[1] As the strata are disjoint, this means that either $X_i = X_j$ or $X_i \subset \overline{X_j} \setminus X_j$.

set is the largest dimension of a stratum. A fundamental result about semi-algebraic sets is that every such set has a semi-algebraic stratification [1, 2].

4. Main result

Our set of interest $\mathcal{H}$ is a polynomial image of $\mathbb{R}^{4I+12}$ (see Section 5.1). Consequently, $\mathcal{H}$ is semi-algebraic and one can speak about its dimension. The main result which we shall establish is the following:

Theorem. *The dimension of* $\mathcal{H}$ *is equal to* $4I + 7$.

We shall split the proof of this theorem into two parts, corresponding to the two inequalities: $\dim \mathcal{H} \leq 4I + 7$ and $\dim \mathcal{H} \geq 4I + 7$. The first inequality has already surfaced in the literature [5], but the derivation of it that we present here is in some aspects new. The second inequality is novel and constitutes the main contribution of the paper.

5. Upper dimension bound

We first show that $\dim \mathcal{H} \leq 4I+7$. With a view to providing some perspective on our main result, we start by presenting a number of weaker bounds on the dimension of $\mathcal{H}$ obtained earlier and only then do we derive the ultimate bound $\dim \mathcal{H} \leq 4I + 7$.

5.1. Initial upper bounds

Let $\mathbf{H}$ be a multi-homography matrix associated with $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{b} \in \mathbb{R}^3$, $\{\mathbf{v}_i\}_{i=1}^I \subset \mathbb{R}^3$ and $\{w_i\}_{i=1}^I \subset \mathbb{R}$. Then, with $\mathbf{a} = \operatorname{vec}(\mathbf{A})$, for each $i = 1, \dots, I$, the ith column $\mathbf{h}_i$ of $\mathbf{H}$ can be written as

$$\mathbf{h}_i = w_i \operatorname{vec}(\mathbf{A}) + \operatorname{vec}(\mathbf{b}\mathbf{v}_i^\mathsf{T}) = w_i \mathbf{a} + (\mathbf{I}_3 \otimes \mathbf{b})\mathbf{v}_i, \tag{5.1}$$

where $\otimes$ denotes Kronecker product [9]. This implies that

$$\mathbf{H} = \mathbf{S}\mathbf{T}, \tag{5.2}$$

where $\mathbf{S}$ is the 9×4 matrix given by

$$\mathbf{S} = [\mathbf{I}_3 \otimes \mathbf{b}, \mathbf{a}]$$

and $\mathbf{T}$ is the $4 \times I$ matrix given by

$$\mathbf{T} = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_I \\ w_1 & \dots & w_I \end{bmatrix}.$$

An immediate consequence of (5.2) is that, whenever $I \geq 4$, $\mathbf{H}$ has rank at most 4. In other words,

$$\mathcal{H} \subset \mathbb{R}_4^{9\times I} \quad \text{for } I \geq 4, \tag{5.3}$$

this being the rank-four constraint mentioned in the Introduction [10] (see also [12]). Here $\mathbb{R}_k^{m\times n}$ denotes the set of real $m \times n$ matrices of rank at most k. It is well known that $\mathbb{R}_k^{m\times n}$ is a $k(m+n-k)$-dimensional variety in $\mathbb{R}^{m\times n}$ [7].

In particular, $\dim \mathbb{R}_4^{9\times I} = 4(9 + I - 4) = 4I + 20$ for $I \geq 4$. Combining this with (5.3) yields $\dim \mathcal{H} \leq 4I + 20$ for $I \geq 4$.

A stronger bound can be obtained by noting explicitly that any multi-homography matrix $\mathbf{H}$ can be naturally expressed in terms of an underlying array of parameters

$$\boldsymbol{\omega} = (\mathbf{A}, \mathbf{b}, \mathbf{v}_1, \dots, \mathbf{v}_I, w_1, \dots, w_I),$$

where $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{b} \in \mathbb{R}^3$, $\{\mathbf{v}_i\}_{i=1}^I \subset \mathbb{R}^3$ and $\{w_i\}_{i=1}^I \subset \mathbb{R}$. More specifically, if $\boldsymbol{\Pi}(\boldsymbol{\omega})$ is the $3 \times 3I$ matrix given by

$$\boldsymbol{\Pi}(\boldsymbol{\omega}) = [\boldsymbol{\Pi}_1(\boldsymbol{\omega}), \dots, \boldsymbol{\Pi}_I(\boldsymbol{\omega})],$$

where

$$\boldsymbol{\Pi}_i(\boldsymbol{\omega}) = w_i \mathbf{A} + \mathbf{b}\mathbf{v}_i^\mathsf{T} \tag{5.4}$$

for each $i = 1, \dots, I$, then

$$\mathbf{H} = r(\boldsymbol{\Pi}(\boldsymbol{\omega})), \tag{5.5}$$

where r denotes the reshaping map

$$[\mathbf{M}_1, \dots, \mathbf{M}_I] \mapsto [\operatorname{vec}(\mathbf{M}_1), \dots, \operatorname{vec}(\mathbf{M}_I)]$$

with $\mathbf{M}_i \in \mathbb{R}^{3\times 3}$ for each $i = 1, \dots, I$. While the array $\boldsymbol{\omega}$ has entries of different types, it can always be reshaped to a length-$(4I + 12)$ vector, for example

$$[\operatorname{vec}(\mathbf{A})^\mathsf{T}, \mathbf{b}^\mathsf{T}, \mathbf{v}_1^\mathsf{T}, \dots, \mathbf{v}_I^\mathsf{T}, w_1, \dots, w_I]^\mathsf{T},$$

and be viewed as an element of $\mathbb{R}^{4I+12}$. Consequently, the set Ω of all arrays $\boldsymbol{\omega}$ as above has dimension $4I + 12$. As (5.5) says that $\mathcal{H}$ is the image of Ω under the composite mapping $r \circ \boldsymbol{\Pi}$ and as $r \circ \boldsymbol{\Pi}$ is smooth, we conclude that $\dim \mathcal{H} \leq 4I + 12$.

This estimate can be further refined to the inequality $\dim \mathcal{H} \leq 4I + 10$ [3]. Indeed, it follows from (5.1) that any multi-homography matrix $\mathbf{H}$ splits as the sum

$$\mathbf{H} = \mathbf{H}' + \mathbf{H}'',$$

where

$$\mathbf{H}' = [w_1 \mathbf{a}, \dots, w_I \mathbf{a}] = \mathbf{a}\mathbf{w}^T, \quad \mathbf{w} = [w_1, \dots, w_I]^\mathsf{T}$$

and

$$\mathbf{H}'' = [(\mathbf{I}_3 \otimes \mathbf{b})\mathbf{v}_1, \dots (\mathbf{I}_3 \otimes \mathbf{b})\mathbf{v}_I] = (\mathbf{I}_3 \otimes \mathbf{b})\mathbf{V}, \quad \mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_I].$$

Clearly, $\mathbf{H}'$ is a rank-one $9 \times I$ matrix. Corresponding to $\mathbf{H}''$, define a $3 \times 3I$ matrix $\mathbf{H}_0''$ by

$$\mathbf{H}_0'' = [\mathbf{b}\mathbf{v}_1^\mathsf{T}, \dots, \mathbf{b}\mathbf{v}_I^\mathsf{T}] = \mathbf{b}[\mathbf{v}_1^\mathsf{T}, \dots, \mathbf{v}_I^\mathsf{T}].$$

The factorisation in the rightmost term shows that $\mathbf{H}_0''$ has rank one. Now, $\mathbf{H}'' = r(\mathbf{H}_0'')$, and so

$$\mathbf{H} = \mathbf{H}' + r(\mathbf{H}_0'').$$

Given that the varieties $\mathbb{R}_1^{9\times I}$ and $\mathbb{R}_1^{3\times 3I}$ to which $\mathbf{H}'$ and $\mathbf{H}_0''$ belong have dimensions $I + 8$ and $3I + 2$, respectively, and that r is smooth, we find that

$$\dim \mathcal{H} \leq (I + 8) + (3I + 2) = 4I + 10.$$

5.2. Ultimate upper bound

A still better, in fact optimal, upper estimate of the dimension of $\mathcal{H}$ is $\dim \mathcal{H} \leq 4I + 7$ [5]. We shall derive it by exploiting the fact there are many different parameter arrays describing one and the same multi-homography matrix. Our derivation will pursue a slightly different path than that taken in [5].

For each matrix

$$\mathbf{C} = \begin{bmatrix} \alpha & 0 & 0 & c_1 \\ 0 & \alpha & 0 & c_2 \\ 0 & 0 & \alpha & c_3 \\ 0 & 0 & 0 & \beta \end{bmatrix}, \tag{5.6}$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and $\mathbf{c} = [c_1, c_2, c_3]^\mathsf{T} \in \mathbb{R}^3$, let $\tau_\mathbf{C}$ be the transformation of Ω into itself given by

$$\begin{aligned} \tau_\mathbf{C}(\boldsymbol{\omega}) = (&\beta\mathbf{A} + \mathbf{b}\mathbf{c}^\mathsf{T}, \alpha\mathbf{b}, \\ &\alpha^{-1}\mathbf{v}_1 - \alpha^{-1}\beta^{-1}\mathbf{c}, \dots, \alpha^{-1}\mathbf{v}_I - \alpha^{-1}\beta^{-1}\mathbf{c}, \\ &\beta^{-1}w_1, \dots, \beta^{-1}w_I). \end{aligned}$$

With the matrix composition as group operation and with the 4×4 identity matrix $\mathbf{I}_4$ as neutral element, the set G of all matrices $\mathbf{C}$ as above is a group. Denote by $\mathrm{Aut}(\Omega)$ the set of all one-to-one transformations of Ω. Under the composition of mappings as group operation and with the identity mapping of Ω as neutral element, $\mathrm{Aut}(\Omega)$ is a group. It is readily verified that the function $\tau \colon \mathbf{C} \mapsto \tau_\mathbf{C}$ maps G into $\mathrm{Aut}(\Omega)$ (so that each $\tau_\mathbf{C}$ is a bijection) and is a *homomorphism*:

$$\tau_\mathbf{C}\tau_{\mathbf{C}'} = \tau_{\mathbf{C}\mathbf{C}'}, \quad \tau_\mathbf{C}^{-1} = \tau_{\mathbf{C}^{-1}}$$

for any $\mathbf{C}, \mathbf{C}' \in G$. A critical property of the $\tau_\mathbf{C}$'s is that each of these transformations leaves all the homography matrices unchanged:

$$\boldsymbol{\Pi}(\tau_\mathbf{C}(\boldsymbol{\omega})) = \boldsymbol{\Pi}(\boldsymbol{\omega})$$

for every $\boldsymbol{\omega} \in \Omega$. Thus the $\tau_\mathbf{C}$'s constitute a group of internal symmetries related to the freedom of choice of parameter arrays. The fact that τ is a homomorphism can be phrased as saying that τ is a representation of G in the *gauge group*. The latter group comprises all transformations γ in $\mathrm{Aut}(\Omega)$ such that $\boldsymbol{\Pi}(\gamma(\boldsymbol{\omega})) = \boldsymbol{\Pi}(\boldsymbol{\omega})$ for each $\boldsymbol{\omega} \in \Omega$. Under the equivalence relation in which $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Omega$ are regarded as equivalent whenever $\boldsymbol{\omega}' = \tau_\mathbf{C}(\boldsymbol{\omega})$ for some $\mathbf{C} \in G$, the set Ω is partitioned into classes of intrinsically equivalent parameter arrays, with each class representing exactly one underlying multi-homography matrix. While these classes can vary in size with changing $\boldsymbol{\omega}$, the majority of them—and this is a crucial observation—can be identified with G and hence have dimension 5. We elaborate on this point and its consequences next.

Let

$$\Omega_1 = \{\boldsymbol{\omega} \in \Omega \mid \mathbf{b} = \mathbf{0}\},$$
$$\Omega_2 = \{\boldsymbol{\omega} \in \Omega \mid \mathbf{b} \neq \mathbf{0},\ w_i = 0 \text{ for each } i = 1, \ldots, I\},$$
$$\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2).$$

Note that each of the above three sets is $\tau_{\mathbf{C}}$-invariant for every $\mathbf{C} \in G$. It is clear that $\mathbf{\Pi}(\Omega_1)$ consists of the matrices of the form $[w_1\mathbf{A}, \ldots, w_I\mathbf{A}]$, whereas $\mathbf{\Pi}(\Omega_2)$ consists of the matrices of the form $[\mathbf{b}\mathbf{v}_1^\mathsf{T}, \ldots, \mathbf{b}\mathbf{v}_I^\mathsf{T}]$. Taking into account that the inverse mapping r^{-1} is smooth (r is clearly one-to-one) and $[w_1\mathbf{A}, \ldots, w_I\mathbf{A}] = r^{-1}(\mathbf{a}\mathbf{w}^\mathsf{T})$, and reusing the argument from the last paragraph of the previous subsection, we conclude that $\dim \mathbf{\Pi}(\Omega_1) \leq I + 8$ and $\dim \mathbf{\Pi}(\Omega_2) \leq 3I + 2$. We shall prove shortly that $\dim \mathbf{\Pi}(\Omega_3) \leq 4I + 7$. Assuming this for now, note that together the last three inequalities imply that

$$\dim \mathbf{\Pi}(\Omega) \leq 4I + 7. \tag{5.7}$$

At this point, observe that $\mathbf{\Pi}(\Omega)$ coincides with $r^{-1}(\mathcal{H})$—see (5.5). Note, moreover, that as r is a one-to-one smooth mapping, r and r^{-1} do not change the dimensions of sets that they transform. Consequently,

$$\dim \mathcal{H} = \dim r^{-1}(\mathcal{H}) = \dim \mathbf{\Pi}(\Omega). \tag{5.8}$$

Combining this with (5.7) yields the desired bound $\dim \mathcal{H} \leq 4I + 7$.

To prove that $\dim \mathbf{\Pi}(\Omega_3) \leq 4I + 7$, it suffices to show that, for each $\boldsymbol{\omega} \in \Omega_3$, the class of $\boldsymbol{\omega}$ under the action of the $\tau_{\mathbf{C}}$'s can be identified with G. Indeed, if this is established, then

$$\begin{aligned} \dim \mathbf{\Pi}(\Omega_3) &\leq \dim \Omega_3 - \dim G \leq \dim \Omega - \dim G \\ &= (4I + 12) - 5 = 4I + 7. \end{aligned}$$

We shall show that the mapping $\mathbf{C} \mapsto \tau_{\mathbf{C}}(\boldsymbol{\omega})$ is one-to-one for each $\boldsymbol{\omega} \in \Omega_3$. It suffices to prove that $\tau_{\mathbf{C}}(\boldsymbol{\omega}) = \boldsymbol{\omega}$ implies $\mathbf{C} = \mathbf{I}_4$ for each $\boldsymbol{\omega} \in \Omega_3$. Take an arbitrary $\boldsymbol{\omega} \in \Omega_3$. Then $\mathbf{b} \neq \mathbf{0}$ and $w_{i_0} \neq 0$ for some $i_0 \in \{1, \ldots, I\}$. If $\tau_{\mathbf{C}}(\boldsymbol{\omega}) = \boldsymbol{\omega}$ holds for some $\mathbf{C}$ as given in (5.6), then $\beta^{-1} w_{i_0} = w_{i_0}$, $\alpha\mathbf{b} = \mathbf{b}$, and $\alpha^{-1}\mathbf{v}_1 - \alpha^{-1}\beta^{-1}\mathbf{c} = \mathbf{v}_1$. The first of these equalities implies that $\beta = 1$, the second implies that $\alpha = 1$, and the third together with $\alpha = \beta = 1$ implies that $\mathbf{c} = \mathbf{0}$. Thus $\mathbf{C} = \mathbf{I}_4$, as desired.

6. Lower dimension bound

Here we show that $\dim \mathcal{H} \geq 4I + 7$. This together with the last result of the previous section will imply that $\dim \mathcal{H} = 4I + 7$ and will finish the proof of our theorem.

6.1. Initial reduction

Let Ω_0 be the set of those $\boldsymbol{\omega}$ in Ω for which

$$\|\mathbf{b}\|^2 = \mathbf{b}^\mathsf{T}\mathbf{b} = 1. \tag{6.1}$$

As pointed out earlier, Ω is essentially identical with the Euclidean space $\mathbb{R}^{4I+12}$. Accordingly, Ω_0 can be viewed as a hypersurface in $\mathbb{R}^{4I+12}$. Consider the restriction $\mathbf{\Pi}|_{\Omega_0}$ of the map $\mathbf{\Pi}$ to Ω_0,

$$\mathbf{\Pi}|_{\Omega_0}\colon \Omega_0 \to \mathbb{R}^{3\times 3I}, \quad \mathbf{\Pi}|_{\Omega_0}(\boldsymbol{\omega}) = \mathbf{\Pi}(\boldsymbol{\omega}), \quad \boldsymbol{\omega} \in \Omega_0.$$

Note that the image of Ω_0 by $\mathbf{\Pi}|_{\Omega_0}$,

$$\mathbf{\Pi}|_{\Omega_0}(\Omega_0) = \mathbf{\Pi}(\Omega_0),$$

is equal to the image $\mathbf{\Pi}(\Omega)$ of Ω by $\mathbf{\Pi}$. Indeed, given $\boldsymbol{\omega} \in \Omega$, the right-hand side of (5.4) does not change if $\boldsymbol{\omega}$ is replaced by $\boldsymbol{\omega}_0 \in \Omega_0$ defined as the modification of $\boldsymbol{\omega}$ in which (i) if $\mathbf{b} \neq \mathbf{0}$, then $\|\mathbf{b}\|^{-1}\mathbf{b}$ is substituted for $\mathbf{b}$ and, for each $i = 1, \dots, I$, $\|\mathbf{b}\|\mathbf{v}_i$ is substituted for $\mathbf{v}_i$, and (ii) if $\mathbf{b} = \mathbf{0}$, then an arbitrary length-3 vector $\mathbf{b}_0$ with $\|\mathbf{b}_0\| = 1$ is substituted for $\mathbf{b}$ and all the $\mathbf{v}_i$'s are taken to be zero, with the rest of the entries of $\boldsymbol{\omega}$ remaining unaltered in either case. Now, in view of (5.8), to complete the argument, it suffices to show that $\dim \mathbf{\Pi}(\Omega_0) \geq 4I + 7$.

Given $\boldsymbol{\omega} \in \Omega$, denote by $\mathrm{d}\mathbf{\Pi}_{\boldsymbol{\omega}}$ the *differential* (or the *linearisation*) of $\mathbf{\Pi}$ at $\boldsymbol{\omega}$. For $\boldsymbol{\omega} \in \Omega_0$, denote by $T_{\boldsymbol{\omega}}(\Omega_0)$ the tangent space of Ω_0 at $\boldsymbol{\omega}$ and by $\mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$ the differential of $\mathbf{\Pi}|_{\Omega_0}$ at $\boldsymbol{\omega}$. When a particular local parametrisation σ for Ω_0 is chosen together with $\mathbf{p} \in \mathbb{R}^{4I+11}$ satisfying $\sigma(\mathbf{p}) = \boldsymbol{\omega}$, $\mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$ can be identified with the Jacobian matrix of the composite mapping $\mathbf{\Pi} \circ \sigma$ at $\mathbf{p}$. As it turns out, the dimension of $\mathbf{\Pi}(\Omega_0)$ is identical with the rank of $\mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$ calculated at any $\boldsymbol{\omega}$ belonging to some *generic* subset of $\mathbf{\Pi}(\Omega_0)$. We shall explain this rather delicate point in the next subsection.

6.2. Regular points

First we recall a few concepts from differential topology, including those of a regular point and a regular value of a smooth mapping. Because our mapping of interest $\mathbf{\Pi}|_{\Omega_0}$ is not locally injective or surjective, we shall use a slightly generalised definition of regular point and regular value.

Given a linear map $\mathbf{A}$, denote by $\operatorname{rank}\mathbf{A}$ and $\operatorname{null}\mathbf{A}$ the *rank* and the *nullity* of $\mathbf{A}$; that is,

$$\operatorname{rank}\mathbf{A} = \dim \mathcal{R}(\mathbf{A}) \quad \text{and} \quad \operatorname{null}\mathbf{A} = \dim \mathcal{N}(\mathbf{A}).$$

Let $f\colon X \to Y$ be a smooth map between smooth manifolds X and Y. Let $\mathrm{r}_{\max}(f)$ be the maximal rank of $\mathrm{d}f_{\mathbf{x}}$ for any $\mathbf{x} \in X$. A point $\mathbf{x} \in X$ is called a *regular point* of f if $\mathrm{d}f_{\mathbf{x}}$ has rank $\mathrm{r}_{\max}(f)$, and is called a *critical point* of f if $\mathrm{d}f_{\mathbf{x}}$ has rank less than $\mathrm{r}_{\max}(f)$. A point $\mathbf{y} \in Y$ is a *regular value* of f if every $\mathbf{x} \in f^{-1}(\{\mathbf{y}\})$ is a regular point; this includes the case where $f^{-1}(\{\mathbf{y}\})$ is empty. Otherwise, $\mathbf{y}$ is called a *critical value* of f. We denote by $\operatorname{Reg}(f)$ the set of regular points of f, and by $\operatorname{Crit}(f)$ the set of critical points of f. With this notation, the set of critical values of f is nothing else but $f(\mathrm{Crit}(f))$, and the set of regular values f coincides with $Y \setminus f(\mathrm{Crit}(f))$.

The principal result of this subsection is the following equality:

$$\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0}) = \dim \mathbf{\Pi}(\Omega_0). \tag{6.2}$$

It reduces the calculation of $\dim \mathbf{\Pi}(\Omega_0)$ to the calculation of $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$.

We start by showing that $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0}) \leq \dim \mathbf{\Pi}(\Omega_0)$. As is known, if $\operatorname{rank} \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}_0} = \mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$ for some $\boldsymbol{\omega}_0 \in \Omega_0$, then $\operatorname{rank} \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} = \mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$ for all $\boldsymbol{\omega}$ in some open neighbourhood of $\boldsymbol{\omega}_0$ in Ω_0 [11, §11.2]. In particular, if $\operatorname{rank} \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} = \mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$ for some $\boldsymbol{\omega}_0 \in \Omega_0$, then $\mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$ has constant rank $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$ for all $\boldsymbol{\omega}$ in a open neighbourhood of $\boldsymbol{\omega}_0$. This property combined with the constant rank theorem [11, Thm. 11.1] guarantees that if $\boldsymbol{\omega} \in \Omega_0$ is such that $\operatorname{rank} \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} = \mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$, then there is an open neighbourhood $U \subset \Omega_0$ of $\boldsymbol{\omega}$ such that $\mathbf{\Pi}(U)$ is a $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$-dimensional regular (embedded) submanifold of $\mathbb{R}^{4I+12}$. It follows that $\mathbf{\Pi}(\Omega_0)$ contains a $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0})$-dimensional submanifold, and hence $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0}) \leq \dim \mathbf{\Pi}(\Omega_0)$.

We now prove that $\mathrm{r}_{\max}(\mathbf{\Pi}|_{\Omega_0}) \geq \dim \mathbf{\Pi}(\Omega_0)$. Let $\{S_i\}_{i \in I}$ be a (finite) semi-algebraic stratification of $\mathbf{\Pi}(\Omega_0)$, with d_i the dimension of S_i for each $i \in I$. Let S_{i_0} be any stratum of $\mathbf{\Pi}(\Omega_0)$ of maximum dimension, i.e.,

$$\dim S_{i_0} = \dim \mathbf{\Pi}(\Omega_0).$$

Let

$$X = \mathbf{\Pi}|_{\Omega_0}^{-1}(S_{i_0}).$$

We claim that X is an open subset of Ω_0.

To establish the claim, we first show that for each $\mathbf{M} \in S_{i_0}$ there is an open set $U_{\mathbf{M}} \subset \mathbb{R}^{3\times 3I}$ containing $\mathbf{M}$ such that

$$U_{\mathbf{M}} \cap \mathbf{\Pi}(\Omega_0) = U_{\mathbf{M}} \cap S_{i_0}. \tag{6.3}$$

Assume the contrary. Then there exists $\mathbf{M} \in S_{i_0}$ such that for every open set $U \subset \mathbb{R}^{3\times 3I}$ containing $\mathbf{M}$, there is $i \neq i_0$ such that $U_{\mathbf{M}} \cap S_i \neq \emptyset$. Consequently, there exists a sequence $\{\mathbf{M}_n\}_{n=1}^{\infty}$ of matrices in $\mathbb{R}^{3\times 3I}$ such that $\lim_{n\to\infty} \mathbf{M}_n = \mathbf{M}$ and, for each positive integer n, $\mathbf{M}_n$ is in S_{i_n} with $i_n \neq i_0$. Since the index set I is finite, we can extract a subsequence $\{\mathbf{M}_{n_k}\}_{k=1}^{\infty}$ from $\{\mathbf{M}_n\}_{n=1}^{\infty}$ such that all the $\mathbf{M}_{n_k}$'s belong to one and the same stratum S_j different from S_{i_0}. Then, clearly, $\mathbf{M}$ is in $\overline{S_j}$, and we see that the set $S_{i_0} \cap \overline{S_j}$, containing $\mathbf{M}$, is non-empty. By the frontier condition **(S2)**, $S_{i_0} \subset S_j$ and $d_{i_0} < d_j$. But this contradicts d_{i_0} being the maximum of all the d_i's.

Having established the existence of $U_{\mathbf{M}}$ satisfying (6.3) for each $\mathbf{M} \in S_{i_0}$, we now note that, by the continuity of $\mathbf{\Pi}|_{\Omega_0}$, $\mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}})$ is an open subset of Ω_0 for each $\mathbf{M} \in S_{i_0}$. Since

$$\mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}}) = \mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}} \cap \mathbf{\Pi}(\Omega_0))$$

and, in view of (6.3),

$$\mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}} \cap \mathbf{\Pi}(\Omega_0)) = \mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}} \cap S_{i_0}),$$

it follows that $\mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}} \cap S_{i_0})$ is an open subset of Ω_0 for each $\mathbf{M} \in S_{i_0}$. But

$$X = \bigcup_{\mathbf{M} \in S_{i_0}} \mathbf{\Pi}|_{\Omega_0}^{-1}(U_{\mathbf{M}} \cap S_{i_0}),$$

and this together with the preceding statement implies that X is an open subset of Ω_0, as claimed.

In particular, X is a smooth manifold in its own right and the restriction $\mathbf{\Pi}|_X$ of $\mathbf{\Pi}$ to X is a smooth map from X to $\mathbb{R}^{3\times 3I}$. Since S_{i_0} is a regular (embedded) submanifold of $\mathbb{R}^{3\times 3I}$, $\mathbf{\Pi}|_X$ induces a smooth map $\tilde{\mathbf{\Pi}}_X : X \to S_{i_0}$ between manifolds [11, Thm. 11.20]. If i denotes the natural embedding of S_{i_0} into $\mathbb{R}^{3\times 3I}$, then $\tilde{\mathbf{\Pi}}_X$ and $\mathbf{\Pi}|_X$ are linked by the relation

$$\mathbf{\Pi}|_X = i \circ \tilde{\mathbf{\Pi}}_X. \tag{6.4}$$

Since, by construction, $\mathbf{\Pi}|_X$ maps X onto S_{i_0}, it follows that also $\tilde{\mathbf{\Pi}}_X$ maps X onto S_{i_0}. By the classical theorem of Sard [6, Chap. 1, §1], the set $\tilde{\mathbf{\Pi}}_X(\mathrm{Crit}(\tilde{\mathbf{\Pi}}_X))$ of critical values of $\tilde{\mathbf{\Pi}}_X$ has $(\dim S_{i_0})$-dimensional measure zero, and, because $\tilde{\mathbf{\Pi}}_X$ is surjective, we have

$$\mathrm{r_{max}}(\tilde{\mathbf{\Pi}}_X) = \dim S_{i_0}. \tag{6.5}$$

In particular, $\mathrm{Reg}(\tilde{\mathbf{\Pi}}_X)$ is non-empty and

$$\mathrm{rank}\, \mathrm{d}(\tilde{\mathbf{\Pi}}_X)_{\boldsymbol{\omega}} = \dim S_{i_0}$$

for each $\boldsymbol{\omega} \in \mathrm{Reg}(\tilde{\mathbf{\Pi}}_X)$. In view of (6.4),

$$\mathrm{d}\mathbf{\Pi}|_X = \mathrm{d}i \cdot \mathrm{d}\tilde{\mathbf{\Pi}}_X$$

by the chain rule, and, as $\mathrm{d}i$ is injective, we have

$$\mathrm{rank}\, \mathrm{d}(\tilde{\mathbf{\Pi}}_X)_{\boldsymbol{\omega}} = \mathrm{rank}\, \mathrm{d}(\mathbf{\Pi}|_X)_{\boldsymbol{\omega}}$$

for every $\boldsymbol{\omega} \in X$, so that

$$\mathrm{r_{max}}(\mathbf{\Pi}|_X) = \mathrm{r_{max}}(\tilde{\mathbf{\Pi}}_X).$$

This equality together with (6.5) implies

$$\mathrm{r_{max}}(\mathbf{\Pi}|_X) = \dim S_{i_0}.$$

Since, obviously, $\mathrm{r_{max}}(\mathbf{\Pi}|_X) \leq \mathrm{r_{max}}(\mathbf{\Pi})$, it follows that

$$\dim \mathbf{\Pi}(\Omega_0) = \dim S_{i_0} \leq \mathrm{r_{max}}(\mathbf{\Pi}),$$

as was to be shown.

6.3. Generic points

By virtue of (6.2), all we need is to estimate from below the rank of $\mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$ at some $\boldsymbol{\omega} \in \mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$. In order to proceed with the actual estimation, we shall first have to be able to exclude points at which our calculations might break down. As it turns out, a systematic procedure for excluding such exceptional points can be devised based on the fact that $\mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$ is a so-called Zariski open subset of Ω_0.

Let $\boldsymbol{R}\colon \Omega \to \mathbb{R}^{3\times 3I} \times \mathbb{R}$ be the mapping defined by

$$\boldsymbol{R}(\boldsymbol{\omega}) = [\boldsymbol{\Pi}(\boldsymbol{\omega}), f(\boldsymbol{\omega})], \quad f(\boldsymbol{\omega}) = \|b\|^2 - 1, \quad \boldsymbol{\omega} \in \Omega.$$

Note that, given $\boldsymbol{\omega} \in \Omega_0$, a vector $\delta\boldsymbol{\omega} \in T_{\boldsymbol{\omega}}(\Omega)$ lies in the subspace $T_{\boldsymbol{\omega}}(\Omega_0) \subset T_{\boldsymbol{\omega}}(\Omega)$ if and only if $\mathrm{d}f_{\boldsymbol{\omega}}(\delta\boldsymbol{\omega}) = 0$. This observation together with the equality

$$\mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}}(\delta\boldsymbol{\omega}) = [\mathrm{d}\boldsymbol{\Pi}_{\boldsymbol{\omega}}(\delta\boldsymbol{\omega}), \mathrm{d}f_{\boldsymbol{\omega}}(\delta\boldsymbol{\omega})]$$

implies that

$$\mathcal{N}(\mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}}) = \mathcal{N}(\mathrm{d}\boldsymbol{\Pi}_{\boldsymbol{\omega}}|_{T_{\boldsymbol{\omega}}(\Omega_0)})$$

for any $\boldsymbol{\omega} \in \Omega_0$. As $\mathrm{d}\boldsymbol{\Pi}_{\boldsymbol{\omega}}|_{T_{\boldsymbol{\omega}}(\Omega_0)} = \mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$ for $\boldsymbol{\omega} \in \Omega_0$, we see that

$$\mathcal{N}(\mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}}) = \mathcal{N}(\mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}})$$

and further that

$$\operatorname{null} \mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}} = \operatorname{null} \mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} \tag{6.6}$$

for any $\boldsymbol{\omega} \in \Omega_0$. We also have

$$\operatorname{rank} \mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} + \operatorname{null} \mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} = \dim T_{\boldsymbol{\omega}}(\Omega_0) \tag{6.7}$$

for any $\boldsymbol{\omega} \in \Omega_0$. At the level of the Jacobian matrices, this is nothing else but an instance of the rank-nullity law of linear algebra saying that the rank and the nullity of a matrix add up to the number of columns of the matrix. Now, by definition, a member $\boldsymbol{\omega}$ of Ω_0 is in $\mathrm{Crit}(\boldsymbol{\Pi}|_{\Omega_0})$ if and only if

$$\operatorname{rank} \mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} < \mathrm{r}_{\max}(\boldsymbol{\Pi}|_{\Omega_0}).$$

Equivalently, in view of (6.7), $\boldsymbol{\omega} \in \Omega_0$ is in $\mathrm{Crit}(\boldsymbol{\Pi}|_{\Omega_0})$ if and only if

$$\operatorname{null} \mathrm{d}(\boldsymbol{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} > \dim T_{\boldsymbol{\omega}}(\Omega_0) - \mathrm{r}_{\max}(\boldsymbol{\Pi}|_{\Omega_0}). \tag{6.8}$$

Note that, in analogy to (6.7), we have

$$\operatorname{null} \mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}} + \operatorname{rank} \mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}} = \dim T_{\boldsymbol{\omega}}(\Omega) = \dim T_{\boldsymbol{\omega}}(\Omega_0) + 1$$

for every $\boldsymbol{\omega} \in \Omega$. This in conjunction with (6.6) and (6.8) implies that $\boldsymbol{\omega} \in \Omega_0$ is in $\mathrm{Crit}(\boldsymbol{\Pi}|_{\Omega_0})$ if and only if

$$\operatorname{rank} \mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}} < \mathrm{r}_{\max}(\boldsymbol{\Pi}|_{\Omega_0}) + 1. \tag{6.9}$$

Choosing standard Cartesian coordinates for Ω and representing each $\mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}}$ by a corresponding Jacobi matrix, we see that (6.9) holds if and only if all the $(\mathrm{r}_{\max}(\boldsymbol{\Pi}|_{\Omega_0})+1) \times (\mathrm{r}_{\max}(\boldsymbol{\Pi}|_{\Omega_0})+1)$ minors of $\mathrm{d}\boldsymbol{R}_{\boldsymbol{\omega}}$ vanish. Therefore the set V of all $\boldsymbol{\omega} \in \Omega$ satisfying (6.9) is algebraic. Moreover, Ω_0 is algebraic as well—in fact, Ω_0 is the product algebraic set $\mathbb{R}^{4I+9} \times \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the two-dimensional unit sphere in $\mathbb{R}^3$. Since $\mathrm{Crit}(\boldsymbol{\Pi}|_{\Omega_0})$ is the intersection of V with Ω_0, it follows that $\mathrm{Crit}(\boldsymbol{\Pi}|_{\Omega_0})$ is a *subvariety* of Ω_0—that is, a set obtained from Ω_0 by imposing additional polynomial equations.

Recall that a variety is called *irreducible* if it cannot be represented as a union of two proper subvarieties. It is a basic fact that a variety $V \subset \mathbb{R}^n$ is irreducible if and only if the following property holds: if the product of two polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ vanishes identically on V, then one of the polynomials vanishes identically on V; in other words, the set of all polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ vanishing identically on V is a *prime ideal* of the ring $\mathbb{R}[x_1, \dots, x_n]$. Since the product of two irreducible varieties is irreducible and since both $\mathbb{R}^{4I+9}$ and $\mathbb{S}^2$ are irreducible (the irreducibility of $\mathbb{R}^n$ for any positive integer n is a standard result which stems from the fact that $\mathbb{R}[x_1, \dots, x_n]$ is an integral domain, and for the irreducibility of $\mathbb{S}^2$ see Appendix A), it follows that Ω_0 is an irreducible variety. Thus $\mathrm{Crit}(\mathbf{\Pi}|_{\Omega_0})$ is a proper subvariety of the irreducible variety Ω_0.

In algebraic geometry, a subvariety of a variety V is alternatively called a *Zariski closed subset* of V. As it turns out, a union of a finite number of a proper Zariski closed subsets of an irreducible variety is always a proper subset. Accordingly, a proper subvariety of an irreducible variety may be considered a "small" subset. A complement of a Zariski closed subset of a variety V is termed a *Zariski open subset* of V. Zariski open subsets of an irreducible variety are "large"—the intersection of any finite number of non-empty Zariski open subsets of an irreducible variety is always non-empty. Using the above terminology, $\mathrm{Crit}(\mathbf{\Pi}|_{\Omega_0})$ is a Zariski closed subset of Ω_0 and as such is "small", and $\mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$ is a Zariski open subset of Ω_0 and hence is "large".

The benefit of identifying $\mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$ as a Zariski open subset of Ω_0 is that one can impose finitely many additional polynomial inequalities of the form $p(\boldsymbol{\omega}) \neq 0$, where p does not vanish identically on Ω_0,[2] to hold on $\mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$ and still obtain a non-empty set. This is so because each inequality $p(\boldsymbol{\omega}) \neq 0$ defines an open Zariski subset of Ω_0, and the final set on which all inequalities hold is the intersection of a finite number of non-empty Zariski open subsets of Ω_0—a non-empty set.

It is customary to say that a property holds *generically* on an irreducible algebraic set V, if it holds on a non-empty Zariski-open subset of V. We shall use this terminology in relation to Ω_0. More specifically, we shall speak about a generic point of Ω_0 as a member of some initially unspecified non-empty Zariski open subset of Ω_0 which is intersected with, or—equivalently—is a subset of, $\mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$. The subset can be made precise *a posteriori* as the aggregate all of whose elements of $\mathrm{Reg}(\mathbf{\Pi}|_{\Omega_0})$ that satisfy all the conditions imposed in the proof.

6.4. Upper nullity bound

Let $\boldsymbol{\omega}$ be a generic point in Ω_0. First note that the dimension of $T_{\boldsymbol{\omega}}(\Omega_0)$ equals the dimension of Ω_0 and this, in view of the constraint (6.1), equals $4I + 11$, one less than the dimension of Ω. This together with (6.7) gives

$$\mathrm{rank}\, \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}} = 4I + 11 - \mathrm{null}\, \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}.$$

Remembering that $\mathrm{d}\mathbf{\Pi}_{\boldsymbol{\omega}}|_{T_{\boldsymbol{\omega}}(\Omega_0)} = \mathrm{d}(\mathbf{\Pi}|_{\Omega_0})_{\boldsymbol{\omega}}$, it is clear that to establish that $\dim \mathbf{\Pi}(\Omega_0) \geq 4I + 7$ we need only show that $\mathrm{null}\, \mathrm{d}\mathbf{\Pi}_{\boldsymbol{\omega}}|_{T_{\boldsymbol{\omega}}(\Omega_0)} \leq 4$.

Let

$$\delta\boldsymbol{\omega} = (\delta\mathbf{A}, \delta\mathbf{b}, \delta\mathbf{v}_1, \dots, \delta\mathbf{v}_I, \delta w_1, \dots, \delta w_I)$$

[2] By the Real Nullstellensatz [1, 2], a polynomial $p(\boldsymbol{\omega})$ vanishes identically on Ω_0 if and only if there exist finitely many polynomials $q_1(\boldsymbol{\omega})$, …, $q_n(\boldsymbol{\omega})$ and a positive integer m such that the polynomial $p^{2m}(\boldsymbol{\omega}) + q_1^2(\boldsymbol{\omega}) + \dots q_n^2(\boldsymbol{\omega})$ is divisible by $\|\mathbf{b}\|^2 - 1$.

be a tangent vector to Ω_0 at $\boldsymbol{\omega}$. In view of (6.1),

$$\mathbf{b}^\top \delta\mathbf{b} = 0. \tag{6.10}$$

For $\delta\boldsymbol{\omega}$ to fall into the null space of $\mathrm{d}\boldsymbol{\Pi}_{\boldsymbol{\omega}}$, it is necessary and sufficient that

$$\mathrm{d}(\boldsymbol{\Pi}_i)_{\boldsymbol{\omega}}(\delta\boldsymbol{\omega}) = \delta w_i \mathbf{A} + w_i \delta\mathbf{A} + \delta\mathbf{b}\mathbf{v}_i^\top + \mathbf{b}\delta\mathbf{v}_i^\top = \mathbf{0} \tag{6.11}$$

for each $i = 1, \dots, I$. Assume that $\delta\boldsymbol{\omega}$ is in $\mathcal{N}(\mathrm{d}\boldsymbol{\Pi}_{\boldsymbol{\omega}})$ so that (6.11) holds. Pre-multiplying (6.11) by $\mathbf{b}^\top$ and using (6.1) and (6.10) yields

$$\delta w_i \mathbf{b}^\top \mathbf{A} + w_i \mathbf{b}^\top \delta\mathbf{A} + \delta\mathbf{v}_i^\top = \mathbf{0}. \tag{6.12}$$

Pre-multiplying in turn this equation by $\mathbf{b}$ and subtracting the resulting equation from (6.11) leads to

$$\delta w_i (\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\mathbf{A} + w_i (\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\delta\mathbf{A} + \delta\mathbf{b}\mathbf{v}_i^\top = \mathbf{0}.$$

The latter formula can be rewritten as

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta w_i \mathbf{A} + w_i \delta\mathbf{A}) + \delta\mathbf{b}\mathbf{v}_i^\top = \mathbf{0}, \tag{6.13}$$

which upon post-multiplying by $\mathbf{v}_i$ gives

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta w_i \mathbf{A} + w_i \delta\mathbf{A})\mathbf{v}_i + \delta\mathbf{b}\|\mathbf{v}_i\|^2 = \mathbf{0}.$$

Hence

$$\delta\mathbf{b} = -(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta w_i \mathbf{A} + w_i \delta\mathbf{A})\|\mathbf{v}_i\|^{-2}\mathbf{v}_i. \tag{6.14}$$

Plugging this expression for $\delta\mathbf{b}$ back into (6.13), we find that

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta w_i \mathbf{A} + w_i \delta\mathbf{A})(\mathbf{I}_3 - \|\mathbf{v}_i\|^{-2}\mathbf{v}_i\mathbf{v}_i^\top) = \mathbf{0}.$$

By virtue of the genericity of $\boldsymbol{\omega}$, we may assume that $w_i \neq 0$ for each $i = 1, \dots, I$, and the above equation can be restated as

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\left(\frac{\delta w_i}{w_i}\mathbf{A} - \delta\mathbf{A}\right)\mathbf{P}_{\mathbf{v}_i}^\perp = \mathbf{0}, \tag{6.15}$$

where

$$\mathbf{P}_{\mathbf{v}_i}^\perp = \mathbf{I}_3 - \|\mathbf{v}_i\|^{-2}\mathbf{v}_i\mathbf{v}_i^\top.$$

Another application of the genericity of $\boldsymbol{\omega}$ ensures that, given a pair i and j of distinct indices, the vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ may be treated as linearly independent with their cross product $\mathbf{v}_i \times \mathbf{v}_j$ non-zero. Since

$$\mathbf{v}_i^\top(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{v}_j^\top(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{0},$$

we have

$$\mathbf{P}_{\mathbf{v}_i}^\perp(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{P}_{\mathbf{v}_j}^\perp(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{v}_i \times \mathbf{v}_j.$$

In view of (6.15),

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\left(\frac{\delta w_i}{w_i}\mathbf{A} - \delta\mathbf{A}\right)(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{0}$$

and

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\left(\frac{\delta w_j}{w_j}\mathbf{A} - \delta\mathbf{A}\right)(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{0}.$$

Subtracting the second of these equations from the first, we obtain

$$\left(\frac{\delta w_i}{w_i} - \frac{\delta w_j}{w_j}\right)(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\mathbf{A}(\mathbf{v}_i \times \mathbf{v}_j) = \mathbf{0}.$$

As, again by the genericity of $\boldsymbol{\omega}$, the vector $(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\mathbf{A}(\mathbf{v}_i \times \mathbf{v}_j)$ may be assumed non-zero, we conclude that

$$\frac{\delta w_i}{w_i} = \frac{\delta w_j}{w_j}.$$

In other words, the $\delta w_i/w_i$'s have a common value. Denote this value by $\delta\lambda$. Then (6.15) can be rewritten as

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta\lambda\mathbf{A} - \delta\mathbf{A})\mathbf{P}^\perp_{\mathbf{v}_i} = \mathbf{0}. \tag{6.16}$$

We now show that in fact

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta\lambda\mathbf{A} - \delta\mathbf{A}) = \mathbf{0}. \tag{6.17}$$

It suffices to prove that

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta\lambda\mathbf{A} - \delta\mathbf{A})\mathbf{x} = \mathbf{0} \tag{6.18}$$

for each length-3 vector $\mathbf{x}$. Choose two linearly independent vectors from amongst the $\mathbf{v}_i$'s, say, $\mathbf{v}_1$ and $\mathbf{v}_2$. As any length-3 vector is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_1 \times \mathbf{v}_2$, (6.18) will be established once it is shown that it holds for $\mathbf{x}$ equal to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_1 \times \mathbf{v}_2$. Since $\mathbf{P}^\perp_{\mathbf{v}_1}(\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{v}_1 \times \mathbf{v}_2$, it follows from (6.16) that

$$\begin{aligned}(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)&(\delta\lambda\mathbf{A} - \delta\mathbf{A})(\mathbf{v}_1 \times \mathbf{v}_2)\\ &= (\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta\lambda\mathbf{A} - \delta\mathbf{A})\mathbf{P}^\perp_{\mathbf{v}_1}(\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{0},\end{aligned}$$

so (6.18) holds in the case $\mathbf{x} = \mathbf{v}_1 \times \mathbf{v}_2$. Now

$$\mathbf{v}_1 = \left(1 - \frac{(\mathbf{v}_1^\top\mathbf{v}_2)^2}{\|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2}\right)^{-1}\left(\frac{\mathbf{v}_2^\top\mathbf{v}_1}{\|\mathbf{v}_2\|^2}\mathbf{P}^\perp_{\mathbf{v}_1}\mathbf{v}_2 + \mathbf{P}^\perp_{\mathbf{v}_2}\mathbf{v}_1\right),$$

as direct verification shows. Using this representation together with (6.16) yields immediately

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta\lambda\mathbf{A} - \delta\mathbf{A})\mathbf{v}_1 = \mathbf{0}.$$

Interchanging the roles of $\mathbf{v}_1$ and $\mathbf{v}_2$ in the above argument leads to

$$(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)(\delta\lambda\mathbf{A} - \delta\mathbf{A})\mathbf{v}_2 = \mathbf{0}.$$

Thus (6.18) also holds in the cases $\mathbf{x} = \mathbf{v}_1$ and $\mathbf{x} = \mathbf{v}_2$.

As an immediate consequence of (6.17), we obtain

$$\begin{aligned}\delta\mathbf{A} &= \mathbf{b}\mathbf{b}^\top\delta\mathbf{A} + (\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\delta\mathbf{A}\\ &= \mathbf{b}\mathbf{b}^\top\delta\mathbf{A} + \delta\lambda(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\mathbf{A}.\end{aligned}$$

Let $\delta\mathbf{c}$ be the length-3 vector defined by $\delta\mathbf{c} = \delta\mathbf{A}\mathbf{b}$. Then

$$\delta\mathbf{A} = \mathbf{b}(\delta\mathbf{c})^\top + \delta\lambda(\mathbf{I}_3 - \mathbf{b}\mathbf{b}^\top)\mathbf{A}, \tag{6.19}$$

expressing $\delta\mathbf{A}$ linearly in terms of $\delta\mathbf{c}$ and $\delta\lambda$. The relation

$$\delta w_i = w_i \delta\lambda \tag{6.20}$$

expresses δw_i linearly in terms of $\delta\lambda$. Now (6.14) in which $\delta\mathbf{A}$ and δw_i are replaced by the right-hand sides of (6.19) and (6.20), respectively, gives an expression for $\delta\mathbf{b}$ that is linear in $\delta\mathbf{c}$ and $\delta\lambda$. Finally, (6.12) rewritten as

$$\delta\mathbf{v}_i = -\delta w_i \mathbf{A}^\mathsf{T}\mathbf{b} - w_i(\delta\mathbf{A})^\mathsf{T}\mathbf{b}$$

and combined with (6.19) and (6.20) as in the previous step gives an expression for $\delta\mathbf{v}_i$ that is linear in $\delta\mathbf{c}$ and $\delta\lambda$. Thus all components of $\delta\boldsymbol{\omega}$ depend linearly on $\delta\mathbf{c}$ and $\delta\lambda$, which shows that the null space of $\mathrm{d}\mathbf{\Pi}_{\boldsymbol{\omega}}|_{T_{\boldsymbol{\omega}}(\Omega_0)}$ is at most four dimensional. This completes the proof of the inequality $\dim \mathbf{\Pi}(\Omega_0) \geq 4I + 7$.

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Appendix A. Irreducibility of the unit sphere

Here we show that, for each positive integer n, the n-dimensional unit sphere

$$\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an irreducible real algebraic variety.

Given a positive integer n, suppose that p_1 and p_2 are two polynomials in $\mathbb{R}[x_1, \ldots, x_{n+1}]$ such that $p_1(\mathbf{x})p_2(\mathbf{x}) = 0$ for each $\mathbf{x} \in \mathbb{S}^n$. We have to show that either $p_1(\mathbf{x}) = 0$ for each $\mathbf{x} \in \mathbb{S}^n$ or $p_2(\mathbf{x}) = 0$ for each $\mathbf{x} \in \mathbb{S}^n$. To this end, we parametrise $\mathbb{S}^n$ less the south pole $[0, \ldots, 0, 1]^\mathsf{T} \in \mathbb{R}^{n+1}$ by $\mathbb{R}^n$ using the inverse of the *stereographic projection* from $[0, \ldots, 0, 1]^\mathsf{T}$. Namely, we assign to each $\mathbf{u} = [u_1, \ldots, u_n]^\mathsf{T}$ the point ($\neq [0, \ldots, 0, 1]^\mathsf{T}$) where the line through $[0, \ldots, 0, 1]^\mathsf{T}$ and $[u_1, \ldots, u_n, 0]^\mathsf{T}$ intersects $\mathbb{S}^n$. The algebraic formula capturing this geometric recipe takes the form

$$x_i = \frac{q_i(\mathbf{u})}{r(\mathbf{u})} \quad (i = 1, \ldots, n+1),$$

where

$$q_i(\mathbf{u}) = \begin{cases} 2u_i, & \text{if } 1 \leq i \leq n, \\ 1 - u_1^2 - \cdots - u_n^2, & \text{if } i = n+1 \end{cases}$$

and

$$r(\mathbf{u}) = 1 + u_1^2 + \cdots + u_n^2.$$

Now note that

$$p_1(q_i/r, \ldots, q_n/r) = r^{-k_1}\tilde{p}_1 \quad \text{and} \quad p_2(q_1/r, \ldots, q_n/r) = r^{-k_2}\tilde{p}_2$$

for some polynomials $\tilde{p}_1$ and $\tilde{p}_2$ in $\mathbb{R}[u_1, \ldots, u_n]$ and some non-negative integers k_1 and k_2. As $r(\mathbf{u}) \neq 0$ for each $\mathbf{u} \in \mathbb{R}^n$, we see that $\tilde{p}_1(\mathbf{u})\tilde{p}_2(\mathbf{u}) = 0$ for each $\mathbf{u} \in \mathbb{R}^n$. Since the set of polynomial functions on $\mathbb{R}^n$ is isomorphic, as a ring, to $\mathbb{R}[u_1, \ldots, u_n]$ and since $\mathbb{R}[u_1, \ldots, u_n]$ is an integral domain, it follows that either $\tilde{p}_1(\mathbf{u}) = 0$ for each $\mathbf{u} \in \mathbb{R}^n$ or $\tilde{p}_1(\mathbf{u}) = 0$ for each $\mathbf{u} \in \mathbb{R}^n$. Consequently, either $p_1(\mathbf{x}) = 0$ for each $\mathbf{x} \in \mathbb{S}^n \setminus \{[0, \ldots, 0, 1]^\mathsf{T}\}$ or $p_2(\mathbf{x}) = 0$ for each $\mathbf{x} \in \mathbb{S}^n \setminus \{[0, \ldots, 0, 1]^\mathsf{T}\}$. Now, by the continuity of polynomials in the usual Euclidean topology and the fact that the closure of $\mathbb{S}^n \setminus \{[0, \ldots, 0, 1]^\mathsf{T}\}$ in the usual topology is equal to $\mathbb{S}^n$ whenever $n \geq 1$, a polynomial which vanishes on $\mathbb{S}^n \setminus \{[0, \ldots, 0, 1]^\mathsf{T}\}$ vanishes on the whole of $\mathbb{S}^n$. This implies that either p_1 or p_2 vanishes identically on $\mathbb{S}^n$. The proof is complete.

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